MODELING FLOWS IN COMPLEX GEOMETRIES: ANALYSIS OF THE PENALIZED LAPLACE AND STOKES OPERATORS AND APPLICATION TO THE NAVIER–STOKES EQUATIONS

ROMAIN NGUYEN VAN YEN, DMITRY KOLOMENSKIY, AND KAI SCHNEIDER

Abstract. Penalization approaches are nowadays commonly employed to solve boundary or initial-boundary value problems. They consist in embedding the original, possibly complex spatial domain inside a bigger domain having a simpler geometry, for example a torus, while keeping the boundary conditions approximately enforced thanks to new terms that are added to the equations [7]. One particular example is the volume penalization method [1] which, inspired by the physical intuition that a solid wall is similar to a vanishingly porous medium, uses the Brinkman-Darcy drag force as penalization term. The main advantage of such penalized equations is that they can be discretized independently of the geometry of the original problem, since the latter has been encoded into the penalization terms. Such a simplifcation permits a massive reduction in solver development time, since it avoids the issues associated to the design and management of the grid, allowing for example the use of simple spectral solvers in Cartesian geometries. The gain becomes even more substantial when the geometry is time-dependent, as in the case of moving obstacles [3], or when fluid-structure interaction is taken into account. We present results of a detailed study [6] of the spectral properties of Laplace and Stokes operators, modifed with a volume penalization term designed to approximate Dirichlet conditions in the limit when the penalization parameter, η, tends to zero. The eigenvalues and eigenfunctions are determined either analytically or numerically as functions of η, both in the continuous case and after applying Fourier or finite difference discretization schemes. For fixed η, we find that only the part of the spectrum corresponding to eigenvalues $\lambda \leq \eta^{-1}$ approaches Dirichlet boundary conditions, while the remainder of the spectrum is made of uncontrolled, spurious wall modes. The penalization error for the controlled eigenfunctions is estimated as a function of λ and η. Surprisingly, in the Stokes case, we show that the eigenfunctions approximately satisfy, with a precision $O(\eta)$, Navier slip boundary conditions with slip length equal to $\sqrt{\eta}$. Moreover, for a given discretization, we show that there exists a value of η, corresponding to a balance between penalization and discretization errors, below which no further gain in precision is achieved. These results shed light on the behavior of volume penalization schemes when solving the Navier-Stokes equations, outline the limitations of the method, and gives indications on how to choose the penalization parameter in practical cases. Possible extensions how to deal with Neumann boundary conditions will also be presented [2]. Finally, different illustrations will be given for vortex-dipole wall interactions [5], flapping insect wings, fluid-structure interaction [4] and a dynamical mixer [2].

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FU Berlin, Germany; Mc Gill, Montreal, Canada; and Aix-Marseille Université, France
E-mail address: kschneid@cmi.univ-mrs.fr